**Retention of Mathematical Knowledge by High School Pupils.**

**Introduction.**

It is a most disconcerting aspect of teaching practice to discover, usually by means of some test, that the mathematical knowledge and ability that one expected to be present in the pupils’ minds after some period of teaching and learning does not seem to be there, that it has vanished, that it has not been retained.

This is especially true when it is felt that teaching and learning has been successful, that pupils seem to have grasped the concepts and are able to use this knowledge and solve problems in the classroom situation. Yet later, it seems as though they have never even been exposed to the ideas.

Barry and Davis (1999) also noted this “heart-breaking” (page 499) situation in connection with exams and asked whether the problem was due to true misconception or just carelessness. Whatever, they noted that it held students back from progressing and this is as true of pupils in school as it is of their undergraduates.

It is clear from the literature that this situation is nothing new. If one reviews early issues of Mathematics in Schools or Mathematics Teaching or any of the many journals devoted to education and mathematics education in particular, one sees that these concerns were as worrying then as they are now. For example Beaumont (1960) discussed the implication that after having received so many shocks in the classroom that we might decide that the children are just not intelligent enough to gain from the work we do with them. Similarly Hutton (1978) wondered whether pupils would gain the concepts given enough time and experience or would some children never be able to gain the (mathematical) concepts. To him it was the big question and it does not seem that we are any further along in answering that question. Indeed Holt (1982) noted the case of a pupil who made very little progress in school and was unable to do basic sums yet was being used by the local bowling league to score matches in top league bowling games. There is also the commonly used example of the Brazilian street children who could perform complex mental calculations in connection with their street market activities and yet were unable to do formal sums in a classroom. (Anderson, Reder & Simon 1997 referencing Carraher, Carraher & Schliemann 1985)

There are many mathematical topics in which the failure to retain knowledge is well attested. For instance mensuration or using the mode, median, mean and range. For instance many teachers will have felt that they have taught areas of plane shapes well, that they have used a variety of different teaching strategies and learning opportunities and that children seem to have been quite happily working out areas and perimeters and in using and applying any formulae they may have learnt. Yet the evidence of testing (Foxman, Martini, Mitchell 1980, TIMSS 1996, Standards at Key Stage 3 Mathematics 2000) shows that often children have very little conception of how to solve even quite straightforward problems. It is as though, as one teacher remarked: “that a mental magnet is suspended above the door to my room, and as soon as the children leave all that they have learnt is sucked out of them.”

It is very clear that if children cannot remember, retain and recall basic mathematical knowledge, formulae, algorithms and concepts, then they are going to have significant difficulty accessing mathematics at the higher levels of attainment (re: levels of attainment in the National Curriculum of England and Wales.)

Since it would seem that the key issue is the inability to retain the mathematical knowledge and procedures that children appear to have learnt it would be appropriate to review the major theories of memory and in particular identify those factors that might seem to have a direct bearing upon retention.

**Theories of Memory.**

After over one hundred years of memory research since the days of Hermann Ebbinghaus (1850-1909), the range and variety of theories of memory and the amount of research that has gone into identifying factors affecting memory performance and ability is extensive. Anderson (2000a) provides an extensive overview and consideration of most aspects of memory research. Norman (1982) remarks that in order to remember, one has to have managed three processes: The acquisition of knowledge, the retention of that knowledge, its subsequent retrieval. Each of these processes is complex and subject to a considerable variety of elements affecting success. Eysenck (1999) also noted that memory theorists needed to concern themselves with the processes that occurred at the time of input and storage and at the time of retrieval and output.

One major theory of memory was that postulated by Atkinson and Shiffrin in 1968 (Eysenck 1999, Anderson 2000a, 2000b) which postulated the existence of a hierarchy of memory stores, in particular the focus was upon the idea of a short-term memory store which maintained information by rehearsal and that the information in it would need to be transferred to a long-term, permanent store (by the mechanism of rehearsal), otherwise the information in short-term memory would decay and be lost or that it would be displaced by other incoming information. The long-term store was essentially unlimited in capacity. They felt that forgetting was due to an inability to retrieve or find the appropriate memory trace (the actual memory as recorded in the brain, also called an engram), rather than the disappearance of the trace from memory. This idea was so well received that many people today still feel (wrongly) that it is the accepted memory model of psychologists.

However in 1972 Craik and Lockhart argued that the memory trace was a result of the analysis carried out by the brain and organism on a stimulus and that the depth to which the stimulus was processed resulted in the strength of the memory trace and hence to its retention. There was no need for a hierarchy of memory stores. They noted that passive rehearsal did not result in better memory in contrast to what was predicted by the theory of Atkinson and Shiffrin. Also the amount of sheer rehearsal made no difference to the efficiency of memory. As an example the BBC saturated the airwaves with the details of new radio frequencies. Less than 25% of the people surveyed had learned the new wavelengths. Simple repetition did not lead to better memories (Anderson 2000a, 2000b). The prime factor is the amount and quality of processing being done with the material. Much research has added to this basic concept and it is widely accepted.

There are many features that must be taken into consideration when considering how mathematical knowledge might be retained. What follows is a concise description of the main issues that might be involved in the retention of mathematical knowledge.

**Acquisition of Memory.**

One factor that aids processing is to do with the relevance of the material, how well it fits in with existing cognitive structures. Craik and Lockart (1972) postulated that more meaningful material is processed faster than less meaningful information and would require less processing. For instance if a pupil has some knowledge of the equivalence of fractions it might be suggested that the process of adding fractions might be more easily processed, than if the pupil had not had any real exposure to fractions. Certainly it should take a shorter time. Often this process is referred to as a “learning curve”. New material seems harder and takes longer to learn than familiar material.

Elaboration is one feature that seems to have an important impact upon the depth of processing. Eysenck (1999) discussed the idea of thinking about the word “clock”. Is it processed as simply a dictionary definition or is it processed in connection with other ideas such as history of clocks, types of clock, clocks that the subject has experience with, including their physical characteristics, makes, colours etc. Anderson (2000b) noted that it does not seem to matter whether the elaborations are generated by the experimenter (teacher) or by the subject (pupil). What is of importance is how well the elaborations relate to the material. For instance semantic elaborations work better than rhymes for remembering words. However in contrast, actively generating a rhyme oneself does lead to better performance – the generation effect (Anderson 2000a). Many teachers are instinctively aware of this in that they develop different types of material and try many different approaches in trying to help pupils develop suitable cognitive structures. In accord with what Anderson noted about the generation effect was the trend towards pupil-centred activity learning, where pupils in working with a mathematical activity might be more enabled to generate their own elaborations of the material therefore leading to better understanding and retention.

Interestingly Anderson, Reder and Simon (1997) noted that it was possible that retention (and transfer) correlated with understanding. If this were the case then tests of retention should indicate understanding. Indeed the problem with Craik and Lockharts’ theory is that it is notoriously difficult to assess the degree of the depth of processing being done upon material and thus assess the strength of the memory trace (Eysenck 1999).

It was observed by Kolers (reported in Anderson 2000b) that subjects could remember more about upside down sentences than they could about normally written sentences. In this case it does not seem that meaningfulness is important, but that because more extensive processing had taken place, the memory trace was stronger. Similarly Hartley (1993) noted that the way material is presented in terms of its format also affected the way that material was retrieved. Material was retrieved in the same way it was presented.

The distinctiveness of an item also seems to play some role in how well it is remembered. For instance we might go regularly to a restaurant, but only remember the occasion when the waiter spilled some food on us. Again good teachers do try to create some excitement or interest in a topic before getting down to the routine of it. For example using a graphical calculator for the first time seems to generate this excitement and interest among pupils. But it is not certain whether this translates into better retention of knowledge about linear functions for instance.

Memory is also strengthened by (meaningful) practice, which has the effect of delaying the onset or degree of forgetting. This may be because of the actual bio-chemical changes occurring in the cortical areas of the brain. Practice has the effect of stimulating the neural pathways more often thus increasing the long-term potentiation (LPA) or responsiveness, of the neurons. (Ashman and Conway 1997). This is akin to scoring a table with a knife; the more the table is scored in the same place the deeper the cut becomes. Anderson (2000a) suggested that the effects of practice are very significant. The proverb “practice makes perfect “ is a mainstay of traditional mathematics teaching and it is clear that practicing procedures does have a beneficial effect. But this has to be qualified, since it is clear that many pupils can practice algorithms quite well, but are unable to use and apply this knowledge in unfamiliar situations. For instance Holt (1982) documents this kind of failure many times and Skemp (1976) used the example of being in a new town and learning the essential routes. If the child is placed in a slightly different part of the town they become lost because the do not have a conceptual picture or map of the town and are thus unable to apply their knowledge of routes to get to where they need to be.

Human memory is highly organised and material that is more organised is better remembered and recalled. (Eysenck 1999). Anderson (2000a) also noted that it is better if the material is hierarchically organised. In fact the brain seems to impose order on material even if it is presented in a random manner. This phenomenon is known as categorical clustering. Eysenck (1999) reports that Weist suggests that this occurs during learning as opposed to at time of retrieval. A study by Arzi, Ben-Zvi and Ganiel (1985) studied the retention of meaningful material (physical science) and also concluded that material organised in a hierarchical sequence was better retained than material taught in a discontinuous array of discrete courses. Mathematics is fortunate as a subject in that it is inherently well organised and logically structured. Yet it is found that pupils often have significant gaps in their comprehension of ideas that were thought to be well understood and practiced. It can be erroneously assumed that pupils have basic concepts that they actually do not have. For instance in 1998 De Bock, Verschaffel and Janssens commented that the lack of use of a paving (using unit squares) strategy in calculating areas of rectangles was surprising given that *“paving is a very easy, intuitive, context bound method, requiring only a little sophisticated formal mathematical knowledge.”* However a year or so later Outhred and Mitchelmore (2000) noted in their research of the use of the array (paving) among primary school children that its use was not intuitively obvious to children. In fact it took some developmental processes for the concept to be acquired. So though the brain might find it better to accept hierarchical material this has to be based upon sound foundations of knowledge and understanding.

**Maintenance of Memory.**

The second vital requirement is that our memories are maintained. Ebbinghaus determined that in general once things had been learned there was a period of initial rapid forgetting, but that this rate slowed down over time. Ebbinghaus was dealing with nonsense syllables rather than the complex requirements of learning mathematics. However most people would find some sense of familiarity with the idea of a retention curve. Many people who learnt a foreign language as a pupil will be aware that they have forgotten words that they know they once knew.

However there is the issue of whether they have actually forgotten the words or that they are simply no longer able to retrieve the words. Norman (1982) made the telling point that it is impossible to know whether we have forgotten something or are simply unable to access it, but the suspicion is that we are unable to access it. Using the same situation of learning a foreign language, it is also apparent that when these same people are presented with the “forgotten” word, there is sometimes a feeling of recognition. Indeed people can be aware that they know something, even if they cannot actually remember or recall it, for example knowing someone’s name or the answer to a quiz question. Pupils often find this when re-presented with a formula.

As another example Nelson (1978) remarked how it was easier for subjects to re-learn items in a list that they had learned two weeks previously, than if they had not learnt them at all, even though on the face of it they seemed to have forgotten the items on the list. Noice and Noice (1997) also observed from anecdotal evidence, that the time for professional actors to re-learn a play done a long time ago and seemingly forgotten was significantly less than the time needed to learn the play from scratch. They also examined the retention of roles they played, over a delayed period of three months. Although the actors had learned and performed new roles in the interim, retention was still very high at 90%. This would undoubtedly be due to the very high level of processing that was involved in learning the words and stage directions, but it was clear that the fall off in retention was less than might have been anticipated, especially with the interference expected from the new roles.

It seems that over long periods of time, memory for general facts, principles and concepts is not significantly affected by the length of time since these things were studied or indeed by the age of the subject (Cohen, Stanhope and Conway 1992). Studies by Bahrick in 1984, showed that the retention of Spanish learned at high school and not used since showed a rapid loss over the first three to five years, but was then stable. He noted that the amount of knowledge retained in this stable period varied with the amount of initial learning and the standard achieved (Cohen, Stanhope and Conway 1992 and Anderson 2000b). Cohen et al (1992) also noted the same principles as applying to the former students they themselves examined. As an example, it is surprising how many adults, especially those taught traditionally, remember the verbal script for Pythagoras’ Theorem, i.e. “The square on the hypotenuse …”

It is also notable from experiments that it does not seem to matter if people intend to learn material or not. For instance Anderson (2000b) talked about how students claimed that it was easier to remember something from a novel that they were not trying to remember rather than from a textbook which they were. This increased facility it was theorised was due to the type of processing going on for the novel. It is much easier to elaborate material for the novel, imagination, characters, plot etc. Anderson makes this telling quote:

 “*The failure of motivation to affect learning can be seen from the viewpoint that people cannot learn what is important to them or from the viewpoint that they cannot avoid learning things that are unimportant.”* (Page 202).

He suggested that the second viewpoint was the more likely. If this were the case then the implication is that people simply have a problem in accessing the memories they have.

Cohen, Kiss and le Voi (1993) discussed the artificiality of laboratory research and how although they were important theoretically, we had to be careful about their application to everyday life. Norman (1982) also noted that sometimes we are unable to remember things we are keen and highly motivated to learn, such as learning a foreign language for when we go on holiday.

Motivation is commonly accepted as being of great importance in learning. Race (Ellington, Percival, Race 1984,1993) put the “Needing/Wanting” element as the driving and pervading force motivating learning in his “ripples” model of learning. Craik and Lockart (1972) identified a reason for this when they noted that the instruction to learn improved memory performance only insofar as it leads the learner to process the material in a more effective manner than if the intention to learn is not there. The learner is more likely to engage in practices that will lead to better retention and recall. (Anderson 2000b). Arnstine suggests that better learning takes place if the pupil finds the experience enjoyable, or at least useful (Arnstine 1996).

Closely connected with motivation is the notion of mathematical anxiety. It is indicated that mathematical anxiety correlates negatively with mathematical achievement (Revak 1997) and that cognition is bound up with emotion especially that of shame associated with rejection. (Ingleton and O’Regan 1988). Holt (1982) makes much mention of this in his book, especially when children refuse to answer or engage with material. Basically he asserts that they are scared/ashamed that they do not understand or know how to do the activity. Sometimes pupils’ relationship with their teacher may inhibit their involvement in the process.

Another factor that affects retention is to do with the meaningfulness of the material. Ebbinghaus focused on nonsense syllables, which showed the characteristic rapid decay in retention. More meaningful material it has been shown in a wide variety of studies shows a significantly reduced loss in retention. (Anderson 2000a) Information that is important also tends to be better remembered. Again it seems clear that many pupils find the whole subject of mathematics highly irrelevant to their daily lives. Mathematics teachers are used to the question: “What’s the point of all this?” Indeed some commentators are wondering what precisely is the point? (Bramall & White 2000, Wiener 2000). If pupils are not finding the subject relevant or meaningful to their lives, is it any wonder that they have problems with remembering and retaining it, or in processing it in such a way as to make access to it more likely?

Retention also seems to be affected by arousal at time of learning. High arousal, such as when it is important to learn the material, leads to better retention. Also time of day has some impact. We are all aware of the Friday afternoon situation, and many teachers have noted that first thing in the day seems to be a better time for learning. I am not aware of any research done into this formally? Material learned at night is better remembered than that learned during the day (Anderson 2000a). This might be a result of the fact that there are less distractions and noise during night time.

A pupil’s cognitive ability must also play some part in their ability to remember material, especially if Craik and Lockart are correct that memory is dependent upon processing ability. In an analysis of the retention of biology material, Adeyemi (1992) noted that pupils who had a high value of “field independence” exhibited a higher degree of retention in comparison with those who exhibited field dependence. Field independent students are those who, for instance, are more able to extract a simple geometric figure from a complex one.

**Retrieval of Memory.**

The third consideration is that memories must be retrievable. Eysenck (1999) reports that Tulving had referred to cue-dependent forgetting (as opposed to memory trace dependent forgetting as discussed above). His encoding specificity principle considered how well the information in the cue fitted the information in the memory trace. Interestingly Anderson (2000a) noted that if the mood at test, the conditions at test and even the physical conditions at test matched those at learning then retrieval of material was better. In particular he reported an extreme example of this where divers were tested as to their ability to recall information when learned in the water or on dry land. Best results were obtained when conditions of learning and test matched! Again this phenomenon is quite often noted in the mathematics classroom. An example of this is reported by Beaumont (1960) who noted that the Gestalt psychologist Wertheimer played havoc with a class by rotating the diagram of a parallelogram when investigating their understanding of the rule for the area of the figure. Vinner (1997) noted that pupils are usually looking for which procedure should be used in solving a problem, i.e. looking through their memory banks for some similar situation that they have studied, rather than in actively engaging with the problem and trying to manipulate it in a way in which the solution may be obtained. As he noted, the most important thing is for pupils to give the answer rather than reflect on whether the answer is reasonable or not! For instance a pupil may simply see a quadratic in a question and simply proceed to solve the equation by means of the carefully remembered formula. It doesn’t matter whether the problem required that or not.

We have already considered elaboration as having a part to play in the creation of memories. Eysenck (1999) also observed that researchers had noted that elaboration of material had a part to play in the retrieval of information. Elaboration seems to provide the mind with additional methods of accessing and recalling the information.

Mathematics teachers have often been exasperated by how some material such as knowing the first ten prime numbers can seemingly have been memorised at the start of a lesson, can be forgotten by the end. This has reference to how active the memory trace (engram) is. In other words memory for things we are still dealing with is good, but if we need to consider other things or are distracted, then the memory trace becomes de-activated and is therefore subject to delay in retrieval. (Loftus discussed in Anderson 2000b). Also memory can also be degraded by the interference of other material that is learnt. These and other interference effects obviously have some impact on retention and recall and this of course is the rule in school situations whereby one lesson on mathematics is followed by another in geography and so on. This would limit the time for any additional processing of material and also cause it to become inactive.

Organised knowledge (semantic memory) may contain millions of items and yet particular memories can be retrieved with ease and very rapidly. Collins and Loftus (as reported in Eysenck 1999) postulated that items are stored according to their semantic relatedness to each other. For instance when the word “dog” is heard then a “spreading activation” among related memory nodes takes place; types of dog, a particular dog, cats, walks, anatomy of dogs etc. Concepts that are more related become more strongly activated and thus more easily retrieved. Anderson’s ACT theory also relies on this basic idea (Anderson 2000b). As a particular example, suppose a child is asked what is the name of a triangle where two of the angles are equal. What might happen is that all words and concepts associated with shapes and the concept of angle are activated. Perhaps the strongest association for the word angle is “acute”, since it is taught at an early age in school. It also happens to be a simpler word than isosceles. So is it any wonder the child simply retorts with the word acute! Vinner (1997) also commented on how often this type of cognitive failure occurred and how it was linked to a failure in pupils to perform true cognitive processes, but instead to carry out pseudo-cognitive processes. Faced with the above situation the pupil can either remain silent or express what crosses their mind, often with little attempt to check internally (reflect on) the answer given.

A common strategy used in accessing memories is to think about associated events. For instance we might try and remember where we left our keys by retracing our steps. Many times information that we are unable to retrieve directly can be inferred from related facts. Most of the time we are not even aware that this is happening. Subjects can also infer and recall material that was not studied or isn’t true! This has profound implications in courts of law for instance. In mathematics education this facility is linked with problem solving ability, but of course the inference of untrue facts is also just as likely.

Anderson (2000a) made an important conjecture with regard to how much need there is for the information to be remembered and recalled. He noted that the more likely the information was required the more likely it would be remembered. Indeed the more often information has been needed in the past the more likely it will be needed again in the future. One can immediately see the relevance of this in mathematics education. How much need is there to remember regularly the formula for solving a quadratic equation compared to the regular need to ring home and friends?

**The Spacing Effect on Memory.**

A final consideration is that of a phenomenon known as the Spacing Effect. This concerns the interval in time between when items are learned. It leads to a consideration of whether learning should take place in one episode (massed practice) or whether it should be spread out over longer periods of time and of more sessions (distributed practice.) Experimental evidence suggests strongly that distributed practice leads to better learning. Anderson (2000a) also noted that in order to have better retention of the material, studying it at spaced intervals is desirable.

Dempster (1988) refers to many studies that have shown the ubiquitousness and dependability of the phenomena and asks why it has not been transferred to the classroom situation. He notes that in many cases two spaced presentations are about twice as effective as two massed presentations. The research also showed that although performance on a test given immediately after massed instruction was better, the material was forgotten faster than for distributed practice.

There were some limits on the effectiveness however. The interval between presentations of data had to be more than or equal to about a day. If the interval was too close then material would be easily retrieved from memory and thus subjects would find themselves bored by the familiar material and find it redundant, whereas the process of re-engaging with the material after some time should prove to be more interesting for the student.

However he also identified areas where massed presentation might be better, for instance in the acquirement of simpler isolated skills, or may not apply to pre-school age children, or in needing to cram for exams.

A study by Revak (1997) showed that distributed practice did seem to have statistically significant advantages over massed practice when applied to the use of homework in raising attainment. Similarly Reynolds and Glaser (1964) used programmed instruction methods to investigate the effect of repetition on retention of biological knowledge. They discovered that variations in repetition had only small effects upon retention, but that spaced review had a significant effect upon the retention of the material.

Groeger (1997) reported a study done on Post Office workers in the UK who were being trained to type postcodes. Those who had one one-hour practice per day learned faster than those who had two one-hour sessions per day and those who had one two-hour session per day. This shows that the interval between practice and the amount of time spent does have observable and different effects.

Interestingly Dineen, Taylor and Stephens (1989) noted that students given daily quizzes showed improvement, though not statistically significant, over students given weekly quizzes.

Krug, Davis and Glover (1990) suggested a theoretical framework to explain the results in terms of a deactivation hypothesis, which was a modification of an earlier full-processing hypothesis developed by Dellarosa and Bourne. In this it was suggested that material is not being subjected to as deep a processing during massed practice since it is held within the working memory at the time it is being used, allowing subjects to skim the content. With distributed practice however the student needs to re-engage with the material thus allowing a renewed opportunity for full processing of the material. In the modification it was suggested that even better processing would take place when the material was fully de-activated from working memory. They then suggested that repeated reading of some material that was spaced far enough apart for de-activation to take place would result in increased retention of the material. Their research showed that this was indeed the case.

A review of research on homework to 1977 by Austin (1979) also showed that generally homework seemed to produce better attainment, although this was by no means conclusive. This would be expected if the spacing effect were significant.

Also it has been identified that review plays a role in the retention of material. Review after instruction seems to consolidate the ideas from that instruction, whereas review that is delayed aids in the relearning of forgotten material (Suydam 1984).

In reviewing the research on the Spacing Effect it is clear that there are a number of elements that need to be carefully identified and considered. For instance there is the spacing between periods of instruction and between periods of review. There is also the issue of repetition of material in contrast with development of material.

**Hypotheses.**

The implications of the spacing effect for mathematics education are obvious. Massed practice of a particular topic might well produce good results immediately after teaching has occurred, but because the material has been processed within working memory, retention of material would be poorer than if the material is dealt with again after it has had time to deactivate. An additional spaced instruction should produce an increased retention of material.

The first hypothesis is therefore that retention of mathematical knowledge should be improved if an additional spaced homework is given as opposed to not being given after a period of instruction on a mathematical topic. It is also proposed to allow two weeks time for full deactivation of the material. This is predicated upon the reasoning that the homework should allow the pupil to review and re-activate the necessary memories, as well as giving the opportunity to the pupil of re-learning any material that was not well learned.

To research this hypothesis a module on the perimeter and area of simple plane shapes was taught to two similar groups of year 8 pupils. Both groups were given a module pre-test and a post-test after instruction. For the experimental group an intervention of an additional homework two weeks after the completion of the module was given, with a retention test given to both groups at the end of a six-week period from the end of the module teaching. No feedback was to be given to pupils on any test or homework.

The second hypothesis is that retention with mathematical knowledge should be improved if during instruction use is made of ICT and at the end of instruction pupils engage with a web based test which would give immediate feedback. This is predicated upon the reasoning that the use of the program and the feedback will increase the depth of processing of the material and that the novelty of the use of a dynamic web based applet and web based assessment will increase the distinctiveness of the material.

To research this hypothesis a module on the use of the mean, median, mode and range was taught to two parallel higher ability year 8 groups. Both groups were to be given module pre-tests. The experimental group was to be given an opportunity to use a dynamic applet showing the difference between the mean and median, and that at the end of the module they were to be given a web based assessment test which would provide them with immediate feedback and an opportunity to improve their scores. After the instruction both groups were to be given post-tests and a retention test six weeks after instruction. No feedback was to be given on the tests.

Before going on to describe the results of the research in detail it will be useful to identify from the research literature what is known about pupils’ conception of perimeters and area and of mean, median, mode and range.

**Pupils’ Understanding of Perimeter and Area.**

Although at first look it might seem that the perimeter and area of simple plane figures: triangles, parallelograms, rectangles and compound L shapes, was a fairly simple topic to cognitively grasp and that the formula and ideas were simple, the research shows that this is not the case.

With regard to testing, the Mathematical Development (1980) report noted that “When given a diagram of a rectangle with sides 24cm and 11cm marked on it, just under 60 per cent of the pupils selected the correct value of the perimeter from the alternatives including the size of the area, which was selected by 25%” and “When told the perimeter of a square was 12cm, nearly 35% of the pupils could derive the area correctly, with 25% squaring the perimeter.”

The TIMSS (1996) report noted on page 53 that for pupils in England only 31.7% of year 8 pupils and 42.6% of year 9 pupils could work out the area of a parallelogram.

The KS3 Mathematics report for 2000 noted that only 30% of pupils working at level 6 were able to calculate the area of a circle. (Page 15) and that many pupils expressed the incorrect belief that the perimeter of a shape is always a larger numerical value than its area.

Baturo and Nason (1996) looked at the understanding student teachers had of area and noted that many of them experienced difficulty and a limited comprehension of the concept of area. This shows up a remarkable aspect. As mathematics educators it is probably almost inconceivable that mature people especially cannot deal with such a basic concept as area. Similarly it is clear that young children also have an inadequate understanding of the concepts (Outhred & Mitchelmore 2000). They quoted large scale studies in the US of both elementary and secondary pupils showing that the problem was universal and not limited to developmental issues. A point made by commentators was that for many students conception of area was limited to rote learning and knowledge of formulae. (Baturo and Nason 1996). For instance when questioned as to why the formula for the area of a triangle has a divide by two, rather than three, students revealed that they had no idea and were surprised that a formula like this could even be questioned. They also commented that the students felt that the process of calculating the area of shapes was something they did in school and had only a vague sense that it was something they might use in the real world.

Reinke (1997) asked some pre-service teachers how they would calculate the perimeter of a certain shape. Their responses revealed that not only were they unsure of their responses in general, but that their concerns were valid. 22% of the students confused methods for calculating areas with that of calculating perimeters. Only 11% were able to identify the procedure needed.

Menon showed more evidence for a systemic failure in the educational process. His study was based upon the conceptual knowledge of students training to be primary teachers in Singapore – a country consistently at the top of mathematical league tables. He showed that once again knowledge about perimeter was based upon a narrowly focussed pre-occupation with formula and standard questions. Most students were unable to calculate the perimeter of an L shape when only given the lengths of the longest sides.

We have already referred to Outhred and Mitchelmore’s (2000) findings that the idea of an area as an array is actually more complex than appears at first sight, but even more disturbing was that Der Bock et al (1998) noted that few pupils were even prepared to consider this strategy, relying instead upon their belief that solving the problem was simply a matter of finding and using the correct formula.

Woodward and Byrd (1983) were amazed and appalled that a common misconception about areas and perimeters was that rectangles with the same area must have the same perimeter. An observation also made as noted by the KS3 Mathematics Report 2000.

Pupils also seem unable to perceive correctly the use of correct units for distinguishing areas and perimeters (Baturo and Nason 1996) and also that they have inadequate conception of accuracy and reasonableness in their answers. (Ernest 1988).

Some research also threw light on the pupils’ use (or lack of) drawings to help in their understanding. (Der Bock et al 1998)

As noted before, Vinner (1997) suggested that what seemed to be happening was that pupils were not engaging with the concepts in a true cognitive way or in a relational manner. (Skemp 1976) Instead he postulated a psuedo-cognitive mode of operation. Essentially the pupil wants to give the correct answer. So for instance in the case of the perimeter of a rectangle where the length or width is given the pupil sees only the numbers. A true cognitive approach would take account of the concept of perimeter, realise that the shape includes two other sides for which measurements are not given and account for these. In the pseudo-cognitive approach, the child is looking for a solution with as little cognitive effort as possible. The pupil recognises, perhaps subconsciously, that simply adding the numbers is too simplistic and thus tries the next most useful idea available, that of multiplying. He called this the minimal effort principle.

It is also clear that pupils seem to just latch onto various formulas. They can learn the rules, but are unable to apply them, they do not have the expert heuristics available to them. (Wilson, Teslow and Taylor 1993)

It is clear then that during tuition, attention must be paid to helping pupils grasp the concepts and ideas that surround area and perimeter, rather than simply focussing on the acquisition of computational skills.

**Pupils’ Understanding of Mean, Mode, Median and Range.**

Similar problems to the understanding of concepts of perimeter and area have also been observed to apply to the pupils’ understanding of average and spread. For instance Strauss and Bichler (1988) observed that children do not spontaneously understand the concepts of the arithmetic mean and certainly not of that of the weighted mean even at quite late levels of development.

They identified various properties of the average and noted that conceptually the idea of summing and dividing seemed to be quite straightforward and simplistic. This idea was also noted by Mokros and Russell (1995) who observed that it seemed to be assumed by educators and textbook writers that pupils who understood the idea of a fair share would understand the idea of average. Their research indicated that pupils did seem to develop quite sophisticated ideas of reasonableness and representativeness of data. That is they could identify typical or atypical values quite readily.

Their review of the literature however showed that the understanding of the mean and of its properties was quite undeveloped even amongst college students. For instance if the average family consisted of 3.2 pupils, what was meant by the .2? They also showed that pupils approached the mean in varying ways. Sometimes for instance it was used as representing the mode, as a midpoint (median) or as simply an algorithm. Some saw it as a measure of reasonableness and only a few saw it as a mathematical point of balance.

Pollatsek, Lima and Well (1981) also recognised a failure amongst college level students to fully comprehend the use of the weighted mean, many having a widespread fallacy that it was allowable to simply add means together. Again they noted that many did not even have a feel for why this was a mistaken idea. They observed that failure seemed linked to a general view that the formula was the only important idea.

It was also clear that for many the idea of a mean is simply that of an abstract computation with no idea of the statistical significance of the measure. They do not know what it represents or how it relates to a data set. (Meyer, Channell 1995). They attempted to help pupils develop their understanding, in particular to the idea that the mean was a point of balance. They noted that even the idea of a balancing beam was one that children had difficulty with. Generally they noted that students preferred the idea of sharing.

In terms of application of the concepts, Watson and Moritz (1999) noted that when comparing data sets it was surprising how few students used the mean, even though they knew how to compute it.

Yet again then it can be seen that the understanding pupils have of a widely taught topic is not as secure as one would hope for. The message is clear: the topic needs to be taught with much attention being paid to conceptual understanding, rather than to simply teaching the algorithms.

**Methodology.**

The research was carried out at Endon High school, Staffordshire. This is an oversubscribed school of about 700 pupils aged 11-16, which according to the latest Ofsted report (Feb 2001) offers very good value for money. It is situated in a semi-rural setting, but a sizeable minority of pupils come from estates on the outskirts of Stoke-on-Trent. In general pupils are motivated and keen to do well and parents are generally supportive. The percentage of pupils obtaining 5 A\*-C GCSE grades ranged from 54-69% over the period 1994-2001.

**Methodology for Mensuration Research.**

**Hypothesis** : Retention of mathematical knowledge will be improved if an additional spaced homework is given two weeks after instruction to allow deactivation of the material in memory.

Four classes out of year 8 (12-13 year olds) were used. These comprised of two parallel higher ability classes in which it would be expected that most pupils would obtain end of keystage 3 (SATs) grades at levels 6 and 7, and two parallel medium ability classes in which it would be expected that most pupils would obtain end of keystage 3 (SATs) grades at levels 4 and 5. The higher ability control group 8A was taught by the head of department, a teacher with over 15 years experience and the middle ability control group – 8s2, was taught by the second in department, a teacher with similar experience. The experimental groups: the higher group 8s1 and the middle ability group 8B were both taught by the researcher, a teacher with 5 years teaching experience.

The method was that all groups were given a 9 question pre-test. This same test was used throughout the experiment. The questions on the test were designed to test basic knowledge of calculating the perimeter and area of rectangles and triangles and L shapes. One question tested the ability to calculate the area of a trapezium and one question tested the ability to calculate the volume of a cuboid. One of the questions tested the ability to recognise and adjust for the use of different units in the perimeter of a triangle question. It was pointed out by the second in department that this was actually an impossible triangle! One question also assessed the ability to recognise if it was possible to calculate the area of the given triangle. The rubric on the paper made clear that it was permissible to write: “can’t be done” as the answer to a question.

Unfortunately due to the constraints of time in the research – an impossibly short time-scale between acceptance of the proposal and the start of the school year, the question paper was never properly assessed and given time it might have been better designed and checked. In retrospect, few questions assessed deeper levels of understanding and perhaps more items could have been added to allow more opportunity for pupils to show understanding in case of calculation errors.

The pupils were then given a period of instruction on the topic. A review of the exercise books shows that in the main pupils in the control groups were given examples with the use of formulae and then practice at solving similar problems. No knowledge of what went on in these classrooms is available. For the experimental groups effort was made to try and encourage deeper thinking about the concepts based upon ideas from the published research. For instance in one lesson squared paper was used and pupils were asked to draw shapes with a given area and calculate the perimeter and vice versa. Areas of irregular shapes were calculated using unit squares, triangles and hexagons in order to get across the concept that area is based upon a known given area. The use of coloured acetates and dissection of figures were used to try and show how areas of more complex shapes might be derived and of course traditional exposition and practice techniques were used. Pupils were also given opportunity to consider some of the more cognitively challenging material used in the research as described in the literature review.

It was felt that the tuition in 8s1 had been particularly successful, but that the tuition in 8B had been less successful, due mainly to the researcher teaching the area of a triangle before the area of a rectangle/parallelogram had been taught – confusion with work done for the 8s1 class! One implication of this is that the details for teaching the module should have been more clearly formulated and designed, especially so that all groups could have been taught exactly the material with the same structure.

It also became apparent that the amount of time that the experimental groups were given for the topic, compared to the control groups was longer.

The groups were then given a post-test, using the same test as for the pre-test. Pupils were not given any feedback on results in the experimental groups, though it may have been possible that the control groups may have been given their scores. Again this was due to unclear guidance given to the other teachers by the author and naiveté in research methods.

After two weeks, and in accord with the hypothesis on the spacing effect, the experimental groups were given a 5-question homework. This was designed to cause the pupils to have to look up their notes and to re-engage with the material. In particular some questions were asked which should have caused them to engage at a deeper level with the material. The schedule unfortunately meant that the homework took place over a half term one-week holiday.

After about a period of six weeks from the post-test, all groups were again given the test as a retention test. The timings were not exact given the differing requirements for individual teachers.

Again after a period of another twelve weeks all pupils were given the same test again. It is assumed that no formal additional teaching on the topic took place for any group over this period, but this cannot be guaranteed for the control groups, since it is possible that as part of teacher’s review strategies and in the teacher’s general teaching practice, reference might well have been made to such topics.

Unfortunately, the control group in the middle ability class (8s2) seemed to have been given a review homework just before the first retention test. Although this invalidates the results for the first retention test, the results have still been included as they provide additional information with regard to the longer term retention. But the results for this group should be treated with caution as no guarantee can be given about additional teaching. This group did not take part in the subsequent work on mean, median, mode and range.

In retrospect no good reason can be given as to why feedback was not given on the results of the tests and homework. At the time it was felt that another variable was being added to the research, but since it is normal and fair practice to give feedback, then this now seems unreasonable, especially as any misconceptions that pupils had could have been cleared up.

For the researcher’s own interest the pupils in 8B were given additional tuition after the final retention test. The aim was to see if their marks could be significantly increased. Quite a lot of time was spent in going through the ideas again and providing many opportunities to practice and revise their skills. After the tuition a test was given without notice. They were then given a homework in which to revise for the test and tested one week later. They were given a final retention test some 8 weeks later. It is important to note that the pupils may well have got used to the same test by now and knew for instance that the question on the area of a triangle with all sides given could not be done (by them)!

**Methodology for Mean, Median, Mode, Range Research.**

**Hypothesis** : Retention of mathematical knowledge will be improved if ICT is used during instruction and that a web based review and assessment test is given at the end of instruction.

For this module two groups were used. This comprised of the control group, the higher ability 8A group taught by the head of mathematics, and the parallel higher ability (8s1) experimental group taught the researcher.

A web based question paper consisting of twenty questions was developed using Dreamweaver to create the page and in using Active Server Pages (ASP) scripts that would write the results directly to an Access database for analysis. Unfortunately it was impossible to get the school’s computer server to work with ASP correctly and so a paper copy of the web page was used to create the question papers. This of course was most disappointing.

For this experiment it was decided to create slightly different versions of the test papers for the post-test and retention test papers. Again due to lack of time and poor preparation by the author the pre-test paper was not fully tested and two questions were found to be incorrect or unusable and these questions were not included in the results for that test.

Some questions in the second test paper used for the post test were amended slightly to allow for the use of a multi-choice selection of answers, rather than allowing the pupils to generate their own response, as it was found from the pre-test that pupils seemed unable to respond succinctly enough to some questions for a reasonable analysis to take place.

The questions were designed to try and help ascertain not only the pupil’s ability to answer standard calculation problems, but to see if they could use the notions to try and recreate data and to work backwards. Two questions were included which were to test their understanding of how the mean and median interrelated and questions on the weighted mean were included as an attempt to try and uncover their understanding of this topic. Multiple-choice questions were used and an attempt was made to try and provide alternatives that pupils might well come up with if they were using the wrong concept. For instance on the weighted means question one alternative was to suggest the mean of the means as an answer.

In terms of tuition the experimental group were given much more tuition than were the control group. In particular it seems as if the control group were given fairly straightforward tuition in terms of calculating the mean, median, mode and range. In contrast the experimental group were given a variety of learning opportunities. The teaching began with a discussion of representativeness. What values seemed typical and which ones atypical in a given scenario. Then an overhead acetate comprising various numbers was flashed onto the screen for a few moments, the pupils then being asked to give various statistics about the data. The intention all the time being to try and focus their minds on the reasons for having to calculate statistics. Pupils were then given some traditional instruction in calculating the various statistics, but all the time effort was made to help the pupils to compare and contrast the need and the suitability of the measures. Pupils were given tuition on how to use a graphical calculator to obtain the statistics, and were given opportunity to analyse and compare contrast two datasets in different situations. Pupils were also given exercises on re-constructing possible datasets given various statistics.

Pupils were also given instruction on using a spreadsheet to calculate statistics and to investigate the effects of changing data and finally they used a web page based computer applet designed by the researcher using the web language Java, which allowed them to dynamically compare and contrast the effect of changing data upon the mean and median.

At the end of instruction pupils were given a web based assessment review test. Although the computer system still didn’t allow their responses to be written to a database, it was possible to use Javascript to dynamically mark their answers and to automatically print out their answered question paper.

The pupils found this to be great fun. They were very impressed and excited by how they could get immediate feedback to their answers. It was quite impossible to stop the pupils discussing their answers with each other, especially since there were only enough terminals for half the class at a time. There were also some technical problems in getting the program running and time was limited. However it was felt that the pupils were getting such a lot of satisfaction out of trying to get all the answers correct that they were left to it, although reservations must be expressed as to how many pupils might simply have just obtained an answer to a particular problem rather than actually working it out. However the original answer sheet (as printed off) was used for analysis, and it is felt that these answers were the result of an individuals honest attempt. All pupils were then given a post-test as discussed.

The pupils in 8s1 were given no feedback or additional teaching as a result of this review.

Although there was no control group for 8B, these pupils were also given the same teaching as for 8s1. However these pupils were not given the computer based review test, but were given full feedback and taught how to obtain the answers on the post-test.

Six weeks later pupils were then given another retention test and another retention test was given some 9 weeks after that. No additional homework or instruction was given in the interim.

**Methodology for Questionnaire Study.**

A questionnaire was constructed (figure 3.9). The questions were gathered from a number of sources and where necessary adapted for use with the year 8 pupils. It was felt that it might be useful for the pupils to have the opportunity to answer similar type questions so as to allow for misunderstandings. The questions were randomised and pupils were told that they should be as honest as possible as it was important to the study that honest answers were given and that the teacher didn’t mind if the pupil thought that he was the worst teacher in the world. The pupils did show honesty!

A mistake in giving clear instruction was made for the pupils in 8A as they were not told to write down their names and no indication that they should do so was put on the questionnaire, therefore no correlation could be performed for their data.

**Results and Analysis of Intervention of Additional Homework on Retention of Mensuration Data.**

The data from the test papers for pupils was adjusted by removing from the analysis all pupils who missed any of the four tests. The numbers of pupils remaining in each class is as follows: 8s1 = 30 pupils, 8A = 22 pupils, 8B = 23 pupils, 8s2 = 23 pupils.

The mean test score for each class was calculated and changed to a percentage simply for easy comparison with other results. Note that there were only 9 questions on the test paper and only one mark was given for each question.

The classes did not do the tests at exactly the same time and so the week numbers that each class did the different tests were calculated and adjusted to allow each class to have the same starting week number.

The results were plotted and are as shown in figure 3.1.

**Figure 3.1** The retention of mensuration knowledge for each class.

It is clear that each class did learn the material to similar degrees. The additional homework for 8s1 and 8B did have some small effect, whereas for 8A there was a slight fall off in retention. 8s2 also showed some improvement, although this is put down to an inadvertent revision given before the first retention test. It can also be seen that the fall off in retention is quite slight at a rate of 0.54% per week for 8B and 0.37% per week for 8s2. This is larger than the 0.05% per week for 8s1 and 0.13% per week for 8A.

This small fall off in retention was unexpected. It was assumed that retention would have fallen off at a much greater rate. Indeed the use of bridging units by QCA is predicated on the assumption that pupils lose knowledge over the six week summer holiday.

A chi squared analysis on the individual pupils was carried out by looking at pupils who regressed, (note: I am using the word “regressed” to mean pupils whose retention was imperfect, whose retention of knowledge was less after the retention test than at the post-test, compared with those that didn’t over the period from the post test to the second retention test). The results are as shown in figure 3.2.

|  |  |  |  |
| --- | --- | --- | --- |
| **Observed(Expected)** | 8s1 | 8A | **Totals** |
|   |  |  |   |
| Pupils who regressed | 10 | 7 | 17 |
|   | **9.81** | **7.19** |   |
| Pupils who didn't | 20 | 15 | 35 |
|   | **20.19** | **14.81** |   |
| **Totals** | 30 | 22 | **52** |
|  |  |  |  |
| **Chi 2 Test** | **0.013** |  |  |
| p-value | 0.9084 |  |  |

**Figure 3.2** Chi squared test results on 8s1 v 8A.

The null hypothesis was that there was no difference between 8s1 and 8A. That is the intervention of the additional homework made no difference to retention. If the p-value is less than 0.05 for a 5% significance level then this would imply a significant difference between the classes. As can be seen there is no significant difference between 8s1 and 8A. (Note there was no point in performing the test for 8B and 8s2 since 8s2 results were compromised by the revision.)

Note that these results should be treated with caution due to the low number of pupils involved.

An analysis was also carried out on the questions which showed regression as shown if figure 3.3.

|  |  |  |  |
| --- | --- | --- | --- |
| **Observed(Expected)** | 8s1 | 8A | **Totals** |
|   |  |  |   |
| Questions which showed regression. | 33 | 23 | 56 |
|   | **32.31** | **23.69** |   |
| Questions which didn't show regression | 237 | 175 | 412 |
|   | **237.69** | **174.31** |   |
| **Totals** | 270 | 198 | **468** |
|  |  |  |  |
| **Chi 2 Test** | **0.040** |  |  |
| p-value | 0.8418 |  |  |

**Figure 3.3** Chi squared test results on questions.

Again it can be seen that there is no significant difference between 8s1 and 8A.

From this analysis a tentative assumption can be made that the additional homework did not significantly affect retention of mensuration knowledge.

Because the graph did show an improvement in results for 8s1 compared with 8A upto the first retention test, a chi squared test was performed on pupils showing improvement from the post-test to the first retention test as shown in figure 3.4.

|  |  |  |  |
| --- | --- | --- | --- |
| **Observed** | 8s1 | 8A | **Totals** |
|   |  |  |   |
| Pupils who improved | 14 | 6 | 20 |
|   | **11.54** | **8.46** |   |
| Pupils who didn't | 16 | 16 | 32 |
|   | **18.46** | **13.54** |   |
| **Totals** | 30 | 22 | **52** |
|  |  |  |  |
| **Chi 2 Test** | **2.017** |  |  |
| p-value | 0.1555 |  |  |

**Figure 3.4** Chi squared test results on pupils showing improvement.

As can be seen there is no significant difference at the 5% level. However the lower p-value is certainly more hopeful and again it should be noted that there is a low number of pupils involved in the test.

Looking at the analysis of questions which showed improvement for 8s1 and 8A as shown in figure 3.5, it can be seen that there is a significant improvement (at the 5% level) in questions for 8s1 against those for 8A. From this it is reasonable to suspect that the intervention of an additional homework did improve results for 8s1. This is to be expected of course.

|  |  |  |  |
| --- | --- | --- | --- |
| **Observed** | 8s1 | 8A | **Totals** |
|   |  |  |   |
| Questions which showed improvement. | 36 | 12 | 48 |
|   | **27.69** | **20.31** |   |
| Questions which didn't show improvement | 234 | 186 | 420 |
|   | **242.31** | **177.69** |   |
| **Totals** | 270 | 198 | **468** |
|  |  |  |  |
| **Chi 2 Test** | **6.564** |  |  |
| p-value | 0.0104 |  |  |

**Figure 3.5** Chi squared test results on questions which showed improvement.

With the 8B class, as already explained I was unhappy with the teaching I had done and decided to re-teach the topic. The results are as shown in figure 3.6:

The first four marks are for the pre-test, the post-test and the two retention tests. Additional teaching was then given and the pupils were then tested with another test given a week later after a revision. A final retention test was given some 8 weeks later. It was pleasing to see that additional learning had taken place and that the revision also made a discernable difference. However the retention fall off after this was steeper than for the earlier part of the study.

**Figure 3.6** Analysis of class 8B with additional teaching.

**Results and Analysis of Intervention of the Use of ICT on Retention of Mean, Median, Mode and Range Knowledge.**

The data from the test papers for the pupils was adjusted by removing from the analysis all pupils who missed any of the four tests. The number of pupils in each class was as follows: 8s1 = 26 pupils, 8A = 27 pupils.

The results for class 8B (23 pupils) are also included although there was no control group to contrast them with.

The mean test score for each class was calculated and changed to a percentage simply for comparison with other results. There were twenty questions on each of the tests except for the pre-test which had only 18 questions counted due to errors in the preparation of the test.

The classes did not do the tests at exactly the same time so the week numbers that each class did the tests were calculated and adjusted to allow each class to have the same starting week number.

The results were plotted and are shown in figure 3.7.

**Figure 3.7** The retention of mean, median, mode range knowledge for each class.

As can be seen each class did learn the material. However it is clear that there is evidence of some kind of reminiscence effect (Swenson no date) for each class. In particular 8A and 8B did not in fact regress in their knowledge, but improved over the period. No reason for this can be postulated except perhaps that the doing of the tests acted as a revision lesson. It is clear that pupils are often keen to exchange ideas on how they did in a test after the event. Perhaps this has acted as additional tuition.

8s1 did show some fall off in knowledge – about 0.08% per week. This again is very slight and again is contrary to expectations.

Since 8A, the control group, did not show any fall off in retention it cannot be ascertained whether the use of ICT had any effect upon the retention of knowledge.

However their does seem to be a larger increase in knowledge for 8s1 over 8A, but as already explained 8s1 had a much longer period of tuition compared with 8A and it cannot be determined if the increase in knowledge is due to the use of ICT or simply a longer teaching period.

However for completeness a chi squared test was carried out for 8s1 against 8A to test the null hypothesis that there was no significant difference between the classes in regard to the regression of knowledge from the post test to the second retention test. This is shown in figure 3.8.

|  |  |  |  |
| --- | --- | --- | --- |
| **Observed(Expected)** | 8s1 | 8A | **Totals** |
|   |  |  |   |
| Pupils who regressed | 11 | 7 | 18 |
|   | **8.83** | **9.17** |   |
| Pupils who didn't | 15 | 20 | 35 |
|   | **17.17** | **17.83** |   |
| **Totals** | 26 | 27 | **53** |
|  |  |  |  |
| **Chi 2 Test** | **1.585** |  |  |
| p-value | 0.208 |  |  |

**Figure 3.8** Chi squared test result on 8s1 v 8A.

It can be seen that there is no significant difference between the classes at the 5% significance level.

**Results and Analysis of the Attitude to Maths Data on Retention of Knowledge.**

The full results of the questionnaire on attitudes to maths and about maths are shown in figure 3.9. As is usual Strongly Agree was coded as +2, Agree was coded as +1, Undecided as 0, Disagree as –1 and Strongly Disagree as –2.

In addition an index value of was calculated for each section. This was done by calculating the mean of the statements with an adjustment being made to ensure that the positiveness of the statement was taken into account, by multiplying by –1.

|  |  |  |  |  |  |
| --- | --- | --- | --- | --- | --- |
|  |   | **All** | **8s1** | **8B** | **8A** |
|  | **Personal feelings about maths.** |  |  |  |  |
| 1 | I am hesitant when doing maths problems as I am afraid I will not be able to do them. | -0.1 | 0.0 | 0.1 | -0.3 |
| 4 | I am sometimes unable to think clearly when doing maths. | 0.6 | 0.9 | 0.3 | 0.7 |
| 7 | Maths makes me feel uneasy and confused. | -0.2 | 0.2 | -0.5 | -0.3 |
| 12 | My maths results are usually low compared to other subjects. | -0.1 | 0.4 | -0.2 | -0.3 |
| 15 | Maths is dull and boring, there is no room for personal opinion. | -0.1 | 0.4 | -0.5 | -0.1 |
| 19 | I feel worried and uncertain when I am trying to do maths. | -0.3 | 0.0 | -0.3 | -0.4 |
| 29 | I don't like being asked questions in maths, I'm scared of being wrong. | 0.3 | 0.5 | 0.3 | 0.2 |
| 38 | I only like maths problems if I have done the same type of problem before. | -0.1 | 0.0 | -0.3 | -0.2 |
|  | Index Value | **0.0** | **-0.3** | **0.1** | **0.1** |
|  | **Attitudes towards maths.** |  |  |  |  |
| 5 | To me, maths is just a load of meaningless squiggles. | -0.7 | -0.1 | -0.8 | -0.9 |
| 10 | I find maths enjoyable and stimulating. | -0.5 | -1.1 | -0.2 | -0.4 |
| 13 | I have never liked maths, it is my most disliked subject. | -0.1 | 0.3 | -0.3 | -0.4 |
| 22 | I think maths is a waste of time. | -1.0 | -0.5 | -1.2 | -1.1 |
| 27 | I have always enjoyed studying maths in school. | -0.4 | -1.0 | -0.1 | -0.3 |
| 33 | There is nothing creative about maths, its just memorising formulas and things. | 0.0 | 0.4 | 0.1 | 0.0 |
| 40 | Maths is interesting and I usually enjoy it. | 0.0 | -0.5 | 0.2 | 0.1 |
|  | Index Value | **0.1** | **-0.4** | **0.3** | **0.3** |
|  | **Feelings about teachers/teaching.** |  |  |  |  |
| 2 | My maths teachers aren't very good at explaining things. | -0.5 | 0.2 | -0.5 | -0.9 |
| 9 | My maths teachers don't understand what they are doing. | -1.2 | -0.7 | -1.4 | -1.4 |
| 24 | Our maths teachers want us to do well | 1.5 | 1.3 | 1.4 | 1.6 |
| 42 | My maths teachers are impatient and too quick. | 0.1 | 0.9 | 0.1 | -0.4 |
|  | Index Value | **0.8** | **0.2** | **0.8** | **1.1** |
|  | **Attitudes about maths as important to society.** |  |  |  |  |
| 8 | Maths is needed in order to keep the world running | 0.2 | -0.2 | 0.2 | 0.4 |
| 18 | Maths is less important to people than art or literature. | -0.5 | -0.4 | -0.7 | -0.5 |
| 25 | Maths is not important in everyday life. | -0.8 | -0.8 | -0.6 | -0.8 |
| 30 | An understanding of maths is required by artists and writers as well as scientists. | 0.3 | 0.4 | 0.5 | 0.0 |
| 35 | It is important for business people to understand maths. | 1.2 | 1.1 | 1.4 | 1.0 |
| 37 | Maths is a worthwhile and necessary subject for all people. | 0.9 | 0.6 | 1.0 | 0.9 |
| 39 | Mathematics is not important for a country or society. | -1.1 | -1.1 | -1.1 | -1.1 |
| 44 | Maths has been important in many areas such as science and technology.  | 1.0 | 0.9 | 1.0 | 1.2 |
|  | Index Value | **0.8** | **0.6** | **0.8** | **0.7** |
|  | **Attitudes about maths as personally important.** |  |  |  |  |
| 11 | I would like to improve my ability and skills in maths. | 1.1 | 0.8 | 1.1 | 1.2 |
| 21 | Knowing maths will help me in the future. | 1.3 | 1.3 | 1.2 | 1.4 |
| 41 | Maths helps develop a person's mind and helps to develop thinking ability. | 1.1 | 1.2 | 1.0 | 1.2 |
| 43 | Now that we have computers there is no need to know much maths. | -0.8 | -0.6 | -0.8 | -1.2 |
|  | Index value | **1.1** | **1.0** | **1.0** | **1.3** |
|  | **Preferences about maths.** |  |  |  |  |
| 3 | I like tasks which allow me to use my imagination. | 0.6 | 0.5 | 0.7 | 0.3 |
| 6 | I like learning new things in maths lessons, rather than doing exercises. | 0.8 | 0.8 | 0.8 | 0.7 |
| 14 | I am interested and happy to use maths outside school. | -0.2 | -0.5 | -0.5 | -0.2 |
| 16 | I like doing investigations in maths. | 0.4 | -0.1 | 0.8 | 0.4 |
| 17 | I find it helpful to discuss maths problems with other people. | 1.2 | 1.1 | 1.4 | 1.1 |
| 20 | Maths problems which are based on real life are more interesting. | 0.6 | 0.7 | 0.6 | 0.4 |
| 23 | Maths is a better subject for girls. | -1.0 | -1.1 | -1.0 | -0.6 |
| 26 | I think that if I knew my tables better, maths would be much easier for me. | 0.6 | 0.5 | 0.5 | 0.7 |
| 31 | I like maths because there is always a correct answer. | -0.2 | -0.5 | 0.1 | -0.5 |
| 32 | Shapes and angles are easier than numbers and algebra. | 0.6 | 0.7 | 1.0 | 0.4 |
| 34 | I enjoy trying harder questions or going into things more deeply. | 0.2 | -0.4 | 0.5 | 0.3 |
| 36 | I find real life problems more difficult than others. | -0.1 | -0.1 | 0.0 | -0.2 |
| 45 | It is pointless to learn maths unless you apply it to real problems. | -0.3 | -0.1 | -0.3 | -0.5 |
| 28 | I hate looking stupid in front of people if I get things wrong. | 1.0 | 1.1 | 1.0 | 0.7 |

**Figure 3.9** Questionnaire results on attitudes to and about mathematics.

The results for 8s1 and 8B pupils were compared with the data for mensuration retention and for mean, median, mode and range retention. For each pupil a retention index was calculated for the post-test – 2nd retention test. This was calculated as 1 – (Post-test mark – Retention test mark)/total marks. A value of 1 means that the knowledge gained was perfectly retained. A value less than 1 means that knowledge was lost, a value of greater than one means that knowledge has improved. No comparison was made for 8A since the results for that class were anonymous – another mistake made in providing clear instructions for other teachers! The results for the combined classes are shown in figure 3.11.

An analysis of correlation indices for each shows that there was no significant correlation between any of the attitudes and the amount of retention for the mensuration results or the amount of retention for the mean, median, mode and range results. Again this is no surprise since the amount of fall off in retention is so small anyway.

As a matter of interest I plotted the overall retention index against the Cognitive Assessment Tests (CAT) mean value that were obtained when the pupils were in the previous year. The results are shown in figure 3.10. As can be seen there seemed to be little correlation between the CAT score and the retention of knowledge.



**Figure 3.10** Comparison of CAT score versus retention index.



**Fig 3.11 Comparison of Questionnaire Index against Mensuration and Mean, Median, Mode, Range Results.**

**Discussion.**

The results from this research are notable in that the expected and significant drop off in retention of mathematical knowledge is very much less than anticipated. In fact it seems that what knowledge pupils do gain they retain very well. It is also remarkable that even for the control groups where more usual methods of teaching were employed and no interventions were performed that the pupils have learnt material and have also retained it very well.

The initial assumption based upon “experience”, upon laboratory experiments and on theory that memory decays quite rapidly do not seem to be valid when applied to the learning and retention of a complex subject like mathematics. This might well have been inferred from the research that was done with the retention of language as reported by Bahrick (1984) as reported in Anderson 2000b).

Since the retention is so good even for the control groups, the interventions of additional homework or of using ICT upon retention are of course going to be small, if not insignificant. What can be hoped for is that these interventions have enabled a better learning of the subject and contributed to a better relational understanding and have created more pathways within memory for the material to be accessed. Of course this research has not investigated these hypotheses.

However, given that the research has not provided evidence as to the significance of these interventions, it has provided some insights into the nature of memory and of learning.

Firstly it is clear that even different methods of teaching have led to significant learning for the pupils. For the mensuration results these seem roughly similar, for the material on mean, median, mode and range the learning achieved by the experimental group seems larger than for the control group. This could be due to the amount of time spent on the topic, the use of ICT during the teaching, the result of the computer based assessment, the use of material attempting to develop deep relational understanding as opposed to surface instrumental understanding, or other factors.

There does also seem to be slight evidence of an increase in learning after the end of instruction. This reminiscence effect has been noted before by researchers (as reported in Swenson - undated).

It was also noted that attitudes about mathematics and to mathematics or the general ability index of pupils does not seem to have much effect upon retention of material either. However this has to be put in the context that little fall off in retention was observed anyway!

So what might explain the lack of expected significant loss in memory and even increase in learning after the instruction period?

Firstly let us review in a simplistic manner what was expected to occur. The pupil learns material in some manner. This creates within the mind a set of neuronal links, which result in the creation of a memory trace of a schema of concepts. As the concepts and schemas are generated various linkages with other concepts and schemas may be occurring in temrs of assimilation and accomadation. The depth of processing carried out creates stronger neuronal pathways and generates additional links for accessing the material. As time continues, the memory trace may weaken due to lack of activation, or the links allowing access to the material may weaken or fail, or other learning may interfere with the traces and links and forgetting occurs.

What is required to maintain memory is for processing to re-occur so that the memory trace is strengthened, or the links to the schema are re-established or re-activated. The theories seem to indicate that in any event the retention of material will drop off according to some power law of forgetting. (Anderson 2000a). That is, the rate of loss will be quite steep at first and then slow down. Certainly this has been the result of numerous experiments from Ebbinghaus onwards.

Since forgetting does not seem to be occurring in this case what might be occurring?

It could be that after instruction further processing of the material is occurring, perhaps not at a level conscious to the pupil. Another possibility us that subsequent learning of new material is causing the brain to re-examine the existing schemas within the mind to ascertain whether they are relevant to the new material (i.e. spreading activation – Eysenck 1999). In this way the old material experiences some renewed processing. For instance, and this is only by means of a possible example, perhaps the pupil working with methods for multiplication. In processing these methods the pupil’s mind checks the various existing schemas for any relevance to the new one. In re-examining the schemas they are reactivated. Perhaps the mind then notices that the schema for finding the area of rectangular figures has some relationship with multiplication, i.e. side times side. In this way additional links are created within the brain, the old schema is strengthened because of the reactivation and perhaps as a result of this activation other schemas are re-examined. At times it may be that this reactivation can be quite strong and that suddenly the person becomes aware of a multitude of links within different schemas giving rise to the effect of insight and understanding – the “eureka!” effect.

Another possibility is that the effect of doing the tests themselves might be acting as a trigger to further processing of the material as described above. What is certainly obvious is that even though no feedback might be given about the test results, pupils often perform their own post mortem with each other about the test, out of sight of the researcher. This will also act as an impetus for additional processing, perhaps allowing misconceptions to be cleared up. As an example it was clear that more and more pupils were answering correctly that the area for a certain triangle (question 5) could not be worked out, or that they were becoming clearer in their mind as to associating the appropriate algorithm to the required statistic. Certainly it sometimes seems that pupils are more willing to be corrected by their trust in the peer group than by teachers!

The frequency of the testing might also have acted at a rate that sufficiently reactivated the schemas. It would be useful to carry out the experiment again, but perform the post-test after a much longer period of time. Although not statistically significant, the effect of the additional homework did seem to have some effect upon the group, and even the inadvertent revision given to the lower ability class 8s2 had a slight observable effect. This of course is related to the spacing effect, but in any event additional research should be carried out to see if there is an actual significant effect in learning.

Finally it could be that given the small sample size and the use of the same pupils for both experiments that what has been observed is a local peculiarity and would not be observed with other samples!

However my belief is that this lack of observable loss in memory retention is a real feature of memory for cognitively challenging topics. Contrary to expectation, pupils do not forget such material. They seem to retain what they learn very well indeed.

**Observations upon the research process and activity.**

As my first attempt at carrying out educational research I would like to note my own personal thoughts on the process itself and upon its value to me as a teacher.

Firstly this whole process has been immensely satisfying and challenging. I feel that my knowledge of the teaching and learning process has been considerably deepened and that I have gained valuable and relevant insights into the process itself and into my own practice. I did not realise that there was so much material, research and experience available about the topic of memory and learning. It is as though I have been given a flashlight to have a look around an old Egyptian tomb – The light flashes upon something that glistens and I get the sense of an outline and a shape, but I know that I am going to have to get closer and inspect it more thoroughly. It is exciting and thrilling to discover the gold and the gems that are to be found. It is also very “dusty” and “sweaty” work! (Literally in the case of looking through periodicals in a library!)

Even at this point I do not feel as though I have thoroughly grasped the concepts of memory and learning yet. My knowledge is very much greater, but the processes that might be involved in creating and maintaining memory are complex and not well understood. There are many differing ideas and theories, some of which are highly technical in description. More time is needed for some of these concepts and ideas to be more thoroughly studied and understood. I am just starting out and I do not presume to have properly represented every idea and concept I have described. I hope I have got them basically right, but I realise that there may be misconceptions and misunderstandings in my presentation. My apologies if this is the case.

In terms of the research I have carried out, I have to admit that I have made many mistakes and errors in carrying it out. Firstly I have not planned the work thoroughly enough. I have had the general ideas in my mind and outlined on paper, but it is now apparent to me that for research of this nature there are very many factors and issues that must be taken into account.

Firstly of course are the issues to do with preparing the test materials themselves. For the mensuration work I should have prepared a larger number of questions and thought a bit more deeply about trying to assess deeper understanding within the questions. The tests for the statistics work was better, but could also have been improved. What was absolutely necessary was for the tests to be fully tested and checked. There were too many errors that should have been picked up. In future I would dry run the tests amongst colleagues and other pupils before going live. Of course time constraints were a major problem, but even so I should have prepared all the material first and tested it before doing the research. Hopefully any errors are negligible in terms of the overall findings, but the errors found in the tests are embarrassing! Of course it is easy after the event to say this since I now see what I need or what could be better, whereas at the time I was working in a darker place!

More thought might also have been given to how I was going to analyse the data. I underestimated the amount of time and effort needed to enter data. I also spent a lot of time in trying to re-arrange the data on the spreadsheet into a form suitable for analysis. Also at times I wasn’t sure what coding I might need to perform analyses, such as coding for types of response to given questions or analysing by question rather than overall. However, I now also appreciate that the analysis can go on for a long time and in different directions from what was originally identified and yet the research has to be kept focussed on the original intention. For instance it was clear to me quite early on that retention was very much better than expected and so the interventions planned were not going to have the impact I was looking for. In this sense it was difficult to keep the research on track since really I felt as though other issues and thoughts needed to be looked at. However the research as planned had to be completed, especially given the timescale.

However even though the interventions have not been as significant to the overall knowledge gained from the research as originally envisaged, the knowledge gained from the research itself has been very valuable.

Another area in which I felt I could have done substantially better was in being even more precisely focussed. With a topic such as this one wants to gain a wide area view of what is happening and certainly at the beginning I had little idea of how things would work out, so my research was coarse and allowed too many factors in to play. Firstly the actual teaching methods and practices I did were variable and wide ranging. There was no detailed plan as to how I was going to teach the topics. I would just try everything and anything to ensure that the pupils were going to learn. My focus was on what would happen after the teaching, but of course I should have also been thinking about how my teaching methods might have an impact upon this. Since there was no plan, the other teachers were also teaching using their own methods and ideas and on different timescales. It is clear to me now that these different methods and times may have an effect upon retention, but I have not controlled this. Also my instructions to myself and to colleagues were not clear. For instance I did not provide instructions on exactly how a test should be administered and what should happen after the test. It is conceivable that pupils might have been given a review of similar types of questions from the test or other feedback that I had not anticipated. I should have been very explicit in what I wanted to happen. However to be contrasted with this is that I am imposing on colleagues and I did not want to control or change their own practice in a forceful or threatening manner. With it being a new school for me as well also had implications for me, i.e. fitting in! I am grateful for the co-operation I experienced.

The interventions were also too coarse and it might have been better if they had been better specified. Should the data have shown loss in retention then I would have needed to have been better able to focus on what feature was having an impact. I’m not sure I could have done this in terms of the research on statistics, was it the teaching, or the use of ICT in the teaching? Which bit of ICT? Was it the applet or the spreadsheet, or was it the computer based assessment?

In some senses I would now regard the research I have done as a fact finding mission for myself in that after doing it I am now in a better position to actually focus the research much more tightly. I needed to explore and to discover how the land lay. Until I actually tried things I had no idea what I really needed to do. Now that I have gone through this exploratory phase I feel as though I could be more specific in what I needed to do, if I was to do it again.

However overall the process has been generally successful and I think valuable and valid. I believe that the results are also reliable and that I will get similar results in future with different pupils and methods.

The whole process has been very enjoyable, interesting and valuable and I will continue to develop my knowledge and insight into these topics and will continue to try to use this to improve my teaching and the pupils’ learning.

**Gordon Moore**

**13 October 2001References**

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